Bachelorarbeit

Perturbative Analysis of the Non-Perturbative Renormalization Group Equation

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Contents

Abstract

In this thesis, the non-perturbative renormalization group (NPRG) is used within the local potential approximation to extract perturbative results and to recover known results from perturbation theory in thermal field theory. The perturbative results for the screening mass and the coupling are recovered up to order g^3 correctly. It is shown that the splitting into two energy scales is not enough to obtain the correct results at higher order of the coupling. First results for the thermal mass at order g^4 and $g^4 \ln g$ are presented and compared to perturbation theory. Some ideas to fix the difference, arising from order g^4 between the perturbative results and the results of the NPRG-formalism, are discussed.

1 Introduction

If a gas is heated up, more and more of the particles become ionized and a plasma is generated, like it happens in our sun. At high temperatures, the velocity of the particles increases so far that we have to deal with relativistic effects, such as particle creation and annihilation. This kind of matter can be found in nature, for example in compact stars or in the early universe, shortly after the big bang. In 2000 we managed to create such a system, called quark gluon plasma (QGP), artificially on earth at the Large Hadron Collider (LHC) at CERN using heavy ion collisions. Since then, this field of research became more and more interesting in the last decade. Thermal field theory describes the thermodynamics of such quantum systems at high temperature. Examples include thermal systems of Quantum chromodynamics (QCD) and Quantum electrodynamics (QED). Thermodynamic observables, like the pressure, can be calculated within this theory. For these calculations, it is quite useful to remember some basics of statistical quantum mechanics, which are presented in chapter 2.

QCD is a theory that describes the behavior of fundamental particles such as quarks and gluons and their interaction, the strong interaction. At very high temperatures and pressures there exists a phase called quark-gluon plasma, a state where quarks and gluons are asymptotically free. To apply this formalism to QCD, we have to handle the non-Abelian QCD Lagrangian. However, much insight can be gained using simpler models. One of these so called "toy-models" is the ϕ^4 - theory, which obtains its name from a quartic interaction term in the potential, $-g^2\phi^4$. The first method to calculate the pressure and the other thermal observables is a perturbative expansion of the path integral. The formalism, which allows us to use the well known methods of quantum field theory, like Feynman diagrams and the path integral, also in the regime of thermal field theory at high temperatures, is explained in chapter 3. In chapter 4, the bad convergent behavior for the perturbative calculations, even for the pressure, are shown. In order to do this, I start with the perturbative expansion of the path integral for the ϕ^4 - theory, which leads to the diagrammatic rules. With this knowledge, the pressure and the screening mass are calculated up to order $O\left(g^3\right)$ of the coupling. In order to avoid infinite values for these quantities, arising due to vacuum fluctuations, I introduce the renormalization scheme, which will be important in the subsequent sections of this thesis. Another method to calculate the expectation values, is the non-pertrubative renormalization group (NPRG), a formalism that takes the whole path integral into account. This formalism is explained in chapter 5, where I also present the derivation of the flow equation, the central equation of the NPRG-formalism.

It is possible to re-extract the perturbative results of the flow equation order by order of the coupling. As shown in chapter 6 and in literature [1], the results match exactly with the corresponding results of perturbation theory up to order $O\left(g^3\right)$. In higher order, it is expected, that the results differ because of the different renormalization schemes. In chapter 7, I try to compute the behavior of the screening mass in higher order of g. I show, that the introduction of an additional energy scale, in order to compute a physical result for the mass, is necessary and discuss some possible ideas to close the gap between the NRPG-formalism and the perturbation theory.

2 Thermodynamic Principles

2.1 Probability

We have to handle two kinds of probability in statistical quantum mechanics. The first one arises from the Heisenberg uncertainty principle. In quantum mechanics we are only able to compute expectation values of operators

$$\left\langle \hat{A} \right\rangle = \langle \Psi | \, \hat{A} \, | \Psi \rangle \tag{1}$$

with the wave function Ψ.

Secondly, completely independent of the quantum level, we have a purely statistical probability: each macroscopic system can be realized by many different microscopic states. So we have to consider many possible quantum states which can be found with a certain probability p_m, $0 < p_m < 1$. If we now want to build the expectation value, we have to average over these possible states too,

$$\left\langle \hat{A} \right\rangle = \sum p_m \langle \Psi_m | \hat{A} | \Psi_m \rangle \,. \tag{2}$$

By defining the statistical operator

$$\hat{\rho} := \sum p_m \, |\Psi_m\rangle \, \langle \Psi_m| \tag{3}$$

we obtain the short hand notation

$$\left\langle \hat{A} \right\rangle = \mathrm{Tr} \left(\hat{\rho} \hat{A} \right) \,. \tag{4}$$

The trace runs over all possible pure quantum states. Since our system has to be at least in one state, the sum over all p_m has to be one, so the trace of our statistical operator has to be one too:

$$\mathrm{Tr}(\hat{\rho}) = 1 \,. \tag{5}$$

In classical statistical mechanics $\rho(p, q)$ is a probability distribution in the $6N$ dimensional phase space, N is the number of particles in the system, and p and q are the canonical coordinates. (In classical mechanics ρ is no operator any more, so the hat is missing.)

The expectation value of a classical observable $A(p, q)$ is given as an integral over the whole phase space:

$$\langle A \rangle = \frac{1}{(2\pi\hbar)^{3N} N!} \int d^{3N}p \, d^{3N}q \, \rho(p, q) A(p, q) \,. \tag{6}$$

2.2 Grand Canonical Ensemble

In statistical physics we want to find a quantity which is constant under a given situation, this quantity is called our thermodynamical potential. The potential should only depend on macroscopic quantities which describe our state. For example, the particle number is a good variable to describe a system when it is constant, otherwise we have to look for another quantity, in this case the chemical potential. To make the potential invariant under the influence of the quantity which we allow to change we perform a Legendre-transformation. Which variables of state we use depends on the physical situation. We distinguish isolated (E, V, N), closed (T, V, N), and open (μ, T, V) systems. To these systems we associate so called ensembles. For example, the microcanonical ensemble is linked to the isolated system, the variables of state are as mentioned before the (inner) energy E, the volume of the system V and the particle number N. All microstates which are compatible with these three macroscopic parameters form realizations of elements of a microcanonical ensemble [3]. Furthermore, there are the canonical ensemble (T, V, N) and the grand canonical ensemble (μ, T, V).

The grand canonical ensemble is used for open systems with a variable particle number N, a non-constant inner energy E but a given volume V and temperature T. In quantum field theory, we allow the creation and annihilation of particles, therefore the number of particles is not a conserved quantity and the grand canonical potential Ω is a good choice. It is related to the pressure of a system:

$$\Omega = -PV = E - TS - \mu N \,. \tag{7}$$

We can guess, that the probability p_m to find a particle in the energy state E_m is proportional to the Boltzmann factor $p_m \propto e^{-\beta(E_m - \mu N)}$, $\beta = 1/T$ in natural units ($k_b = 1$). So the statistical operator becomes

$$\hat{\rho} \propto e^{-\beta\left(\hat{H} - \mu \hat{N}\right)} \,. \tag{8}$$

$\hat{H}$ is the Hamilton operator. From now on I will drop the hats for simplicity. The exponential of an operator is defined via its series expansion. For an exact derivation of this relation see for instance [7].

To guarantee the normalization $\mathrm{Tr}(\rho) = 1$ we divide the Boltzmann operator by the partition function

$$Z = \mathrm{Tr}\left(e^{-\beta(H - \mu N)}\right) \,. \tag{9}$$

The partition function is a quite powerful tool, it allows us to compute all important quantities like $\Omega = -T \ln(Z)$. The expectation value of an operator now becomes

$$\langle A \rangle = \frac{\mathrm{Tr}\left(A e^{-\beta(H - \mu N)}\right)}{Z} \,. \tag{10}$$

3 From Quantum Field Theory to Thermal Field Theory

In 1932, Felix Bloch presented his idea, that the expectation value of a thermal ensemble can be written as the expectation value of an ordinary quantum field theory, using the Matsubara imaginary time formalism.

3.1 The Path Integral in Quantum Mechanics

In the Heisenberg picture the operators of quantum mechanics are time dependent and they are linked to the operators in the Schrödinger picture via

$$Q(t) = e^{iHt} Q e^{-iHt}. \tag{11}$$

An eigenstate of $Q(t)$ can be expressed as

$$Q(t) \left| q, t \right\rangle = q \left| q, t \right\rangle \tag{12}$$

with eigenvalue q. The time dependent eigenstates can be written with the time evolution operator:

$$\left| q, t \right\rangle = e^{iHt} \left| q \right\rangle \, . \tag{13}$$

With these definitions we can write the transition amplitude in a very simple way:

$$\left\langle q'', t'' \right| q', t' \rangle \, . \tag{14}$$

This transition amplitude can be evaluated in a very elegant way using the Feynman path integral formalism. The amplitude then becomes

$$\left\langle q'', t'' \right| q', t' \rangle = \int \mathcal{D}q \, \exp\left[i \int_{t'}^{t''} dt \, L(\dot{q}(t), q(t)) \right] = \int \mathcal{D}q \, \exp(iS) \tag{15}$$

with the Lagrangian function $L(\dot{q}(t), q(t))$ and the action [8]

$$S = \int dt \, L \, . \tag{16}$$

$\mathcal{D}q$ should denote an integration over all possible paths in space, a procedure which is mathematically not understood completely. To be more exact we would have to consider a functional integral over the conjugate momentum too, but in many cases this integral can be solved analytically using Gaussian integrals. The result is just a constant which is taken in account in the integral measure $\mathcal{D}q$. For a more exact derivation and explanation to the path integral in quantum mechanics see for instance [8] or [5].

3.2 The Path Integral in Quantum Field Theory

In quantum field theory we want to allow the creation and annihilation of particles. An operator which corresponds to the addition of a particle or antiparticle with certain quantum number clearly has to depend on these numbers and a position in space time. Therefore we have to use operatorvalued fields in relativistic quantum mechanics. The Hamiltonian depends on the fields and their conjugate momentum and therefore also has a spatial dependence. This is why we are working with the Hamiltonian or Lagrangian density from now, denoted by $\mathcal{H}$ and $\mathcal{L}$. The eigenstates of the field operator are denoted by $|\phi\rangle$ and fulfill the eigenvalue equation

$$\hat{\phi}(\mathbf{x}, 0) |\phi\rangle = \phi(\mathbf{x}) |\phi\rangle \ . \tag{17}$$

If we suppose that the system is at $t = 0$ in a state $|\phi_a\rangle$ and evolves with a non time-dependent Hamiltonian H, the transition amplitude can be written as

$$\langle\phi_b| e^{-iHt_f} |\phi_a\rangle \tag{18}$$

and calculated with the functional integral:

$$\langle\phi_a| e^{-iHt_f} |\phi_a\rangle = \int \mathcal{D}\phi \, \exp\left(i \int d^4x \, \mathcal{L}\right) . \tag{19}$$

Note that we need to integrate the Lagrangian density over four dimensional spacetime to obtain the Lagrangian function (integration over the spatial components) and then the action S. If we add a source term to our Lagrangian, we are able to obtain the expectation value of the field per functional derivation. The source term is proportional to the field:

$$\langle\phi_a| e^{-iHt_f} |\phi_a\rangle_J = \int \mathcal{D}\phi \, \exp\left(i \int d^4x \, [\mathcal{L}_0 + J\phi]\right) . \tag{20}$$

J has to be understood as a current density so it is also integrated over space-time.

3.3 The Path Integral in Thermal Field Theory

Now we want to switch to thermal field theory with the goal to calculate the partition function Z using functional integration. Remember Eq. (9):

$$Z = \mathrm{Tr}(e^{-\beta H}) = \sum_a \int d\phi_a \, \langle\phi_a| e^{-\beta H} |\phi_a\rangle \ . \tag{21}$$

The trace operation is written as a sum over all states. This formula already looks very similar to the left-hand side of Eq. (19). The next step is to switch to the imaginary time formalism, this means we do the substitution $it = \tau$, $dt = \frac{d\tau}{i}$ which cancels the i in the exponent. The only difference between Eqs. (19) and (21) is the integration variable in front of the Hamiltonian. If we integrate from 0 to β instead of t_f, which means that we associate the time with the inverse temperature, there is mathematically no difference any more and we can use the well known path integral formalism. This leads us to the important formula:

$$Z = N \int_{periodic} \mathcal{D}\phi \, \exp\left(\int_0^\beta d\tau \int dx^3 \, \mathcal{L} \right) = N \int_{periodic} \mathcal{D}\phi \, e^S \, . \tag{22}$$

N is just a normalization factor. This result is just correct for a neutral scalar field, i.e. we set the chemical potential $\mu = 0$ because it is not important for the further calculations in this work. The term periodic means that the field ϕ has to fulfill the constraints

$$\phi(\mathbf{x}, 0) = \phi(\mathbf{x},\beta) \, . \tag{23}$$

This is because of the trace operation: in the quantum mechanical case the transition amplitude can be calculated for any states, but the trace means that we have to build the sum of the diagonal elements of the transition matrix, so the two states have to be the same.

Analogously to Eq. (10) we can obtain a formula to calculate an expectation value of the field in thermal field theory:

$$\langle \phi \rangle = \frac{1}{Z} \int \mathcal{D}\phi \, \phi \cdot e^S \, . \tag{24}$$

4 Perturbation Expansion in Scalar Field Theory

4.1 Neutral Scalar Field

The Lagrangian density for a neutral scalar ϕ^4-theory (this means that there are no conserved charges) differs from the free Lagrangian by the interaction term, which gives the theory its name:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - g^2 \phi^4 \tag{25}$$

with $\phi = \phi(x) = \phi(\mathbf{x}, \tau)$. g^2 is known as the coupling constant. This theory is often called a "toy-theory". More physical theories like QCD and QED are much more complicated and not considered in this work.

4.2 Perturbation Expansion

As mentioned earlier, quadratic terms in the Lagrangian lead to Gaussian integrals, which can be solved analytically. If there are terms of higher order, like in ϕ^4-theory, the analytical method breaks down. One possibility is an approximation technique, using perturbation expansion of the path integral and Feynman diagrams for solving the appearing expressions. Other possibilities are Monte-Carlo methods or the NPRG-formalism, which is treated later.

The action in the functional integral of the partition function Eq. (22) can be decomposed in a term quadratic in the field S_0 and an interaction term S_I:

$$S = S_0 + S_I \,. \tag{26}$$

Now we expand the interaction part in a power series

$$Z = N' \int \mathcal{D}\phi e^{S_0} \sum_{l=0}^{\infty} \frac{1}{l!} S_I^l \tag{27}$$

and take the logarithm on both sides, which separates the interaction part completely from the ideal gas contribution:

$$\begin{aligned} \ln(Z) &= \ln\left(N' \int \mathcal{D}\phi\, e^{S_0} \left[1 + \sum_{l=1}^{\infty} \frac{1}{l!} \frac{\int \mathcal{D}\phi\, e^{S_0} S_I^l}{\int \mathcal{D}\phi\, e^{S_0}} \right] \right) \\ &= \ln\left(N' \int \mathcal{D}\phi\, e^{S_0} \right) + \ln\left(1 + \sum_{l=1}^{\infty} \frac{1}{l!} \frac{\int \mathcal{D}\phi\, e^{S_0} S_I^l}{\int \mathcal{D}\phi\, e^{S_0}} \right) \\ &= \ln Z_0 + \ln Z_I \,. \end{aligned} \tag{28}$$

In this calculation we separate the $l = 0$ part of the sum and use the rules for the sums of logarithms. The normalization constant N' is totally irrelevant since it cancels out in the interaction part, which we want to calculate, and is therefore dropped from now on. The relevant quantity to compute (remember $\Omega \propto \ln Z$) is the value of S_I in an arbitrary integral power and averaged over the unperturbed ensemble [6]:

$$\langle S_I^l \rangle_0 = \frac{\int \mathcal{D}\phi \, e^{S_0} S_I^l}{\int \mathcal{D}\phi \, e^{S_0}} \,. \tag{29}$$

4.3 Matsubara Frequencies

In the next section the Fourier transformed field $\phi(K)$ is needed, this is why the concept of Matsubara frequencies has to be introduced. Capital letters will denote the four-dimensional quantities, for example $K = (k_0, \mathbf{k})$, $X = (t, \mathbf{x}) = (-i\tau, \mathbf{x})$ and the Minkowski scalar product will be denoted by $K \cdot X = k_0 x_0 - \mathbf{k} \cdot \mathbf{x}$ using the Minkowski metric diag$(1, -1, -1, -1)$.

If we perform the Fourier transformation of the field, we have to remember the periodicity constraint Eq. (23). A periodic function yields a discrete Fourier transformation, this means:

$$\phi(X) = \frac{1}{\sqrt{TV}} \sum_K e^{-iK\cdot X} \phi(K) = \frac{1}{\sqrt{TV}} \sum_{n,\mathbf{k}} e^{i(\omega_n \tau + \mathbf{k}\cdot\mathbf{x})} \phi(K) \,. \tag{30}$$

In the thermodynamic limit the discrete sum over $\mathbf{k}$ becomes an integral, but the sum over the ω_n stays discrete. To fulfill the periodicity constraints we need

$$e^{i\cdot 0\cdot\tau} = 1 = e^{i\omega_n \beta} \tag{31}$$

which is equivalent to

$$\omega_n = 2\pi n T \tag{32}$$

where ω_n are called "Matsubara frequencies" [7].

4.4 Diagrammatic Rules for ϕ^4- Theory

Evaluating Eq. (28) turns out to be more complicated than it looks at first sight, and we need to use the technique of Feynman diagrams. To evaluate the Feynman rules in ϕ^4-theory, we take a look at the first order correction to $\ln Z_0$:

$$\ln Z_1 = \frac{-g^2 \int d\tau \int d^3x \int \mathcal{D}\phi \, e^{S_0} \phi^4 \left(\mathbf{x}, \tau\right)}{\int \mathcal{D}\phi \, e^{S_0}} \tag{33}$$

where the ϕ^4 - term comes from the interaction part of the Lagrangian:

$$\mathcal{L}_I = -g^2 \phi^4 \,. \tag{34}$$

Note that the first order correction is already of order g^2 since we have written the $\mathcal{L}_I$ term for later simplification already as g^2. Sometimes the ϕ^4- part of the Lagrangian is written as $\mathcal{L}_I = -\frac{\lambda}{4!}\phi^4$, where the couplings are related by $g^2 = \frac{\lambda}{4!}$. In this work we use the definition of Eq. (34).

If we now insert the Fourier transformed field Eq. (30) into Eq. (33), we obtain (for more

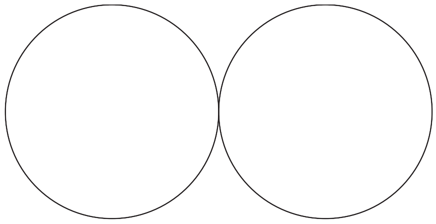


Figure 1: First order correction to the pressure $\ln Z_1$.

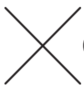
details see [6]) the following expression after a lengthy calculation:

$$\ln Z_1 = -3g^2 \beta V \left(T \sum_n \int \frac{d^3 p}{(2\pi)^3} \mathcal{D}_0 \left(\omega_n, \mathbf{p} \right) \right)^2 \tag{35}$$

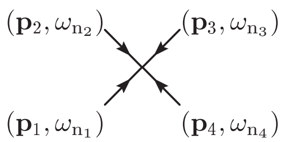
where we have defined the bare propagator:

$$\mathcal{D}_0 \left(\omega_n, \mathbf{p} \right) = \frac{-1}{\omega_n^2 + \mathbf{p}^2 + m^2} \, . \tag{36}$$

Now think of the $\phi^4(\mathbf{x}, \tau)$ term as a diagram. We have to draw a cross with four arms raising from the fourth power of ϕ^4 with the vertex at the space point $(\mathbf{x}, \tau)$:

$\phi^4(\mathbf{x}, \tau)$:  $(\mathbf{x}, \tau)$. (37)

In momentum space, this means after doing the Fourier transformation of every field, we can label each line with the in- or outgoing momentum and the corresponding frequency:



(38)

An important fact in deriving Eq. (35) is that the functional integral vanishes unless for example $n_3 = -n_1$, $\mathbf{p}_3 = -\mathbf{p}_1$, $n_4 = -n_2$ and all the other possible permutations. This means that only so called "bubble diagrams" contribute to the thermal pressure. We just have to connect the ends in pairs and because there are three ways to do this, we obtain, as we have already seen in the calculation a combinatoric factor of 3.

So we can guess the diagrammatic rules for the ϕ^4-theory. The vertex is the point of interaction, so we associate a factor $-g^2$ and for each line we simply write down a factor $\mathcal{D}_0(\omega_n, \mathbf{p})$. Here we can see why $\mathcal{D}_0$ is called a propagator: It moves ("propagates") a particle or a field from one point in spacetime to another. If we have a closed line we have to integrate over it and multiply it with the Temperature T:

$$T \cdot \sum_n \int \frac{d^3p}{(2\pi)^3} \mathcal{D}_0(\omega_n, \mathbf{p})\,. \tag{39}$$

Of course it is possible that one closed line consists of more than one propagator, for example if we attach a second loop to an existing one, so this is just a example.

At second order we would recognize that all not connected diagrams drop out, for a proof look again at [6], p 37.

To sum up, we have obtained the following rules for drawing and calculating the diagrams at order N of the coupling (which is equal to the number of vertices in the diagram):

1. Draw all connected diagrams.

2. Determine the combinatoric factor for each diagram.

3. Include a factor $\mathcal{D}_0(\omega_n, \mathbf{p})$ for each line.

4. Multiply with T and perform the integral and the Matsubara sum for each loop.

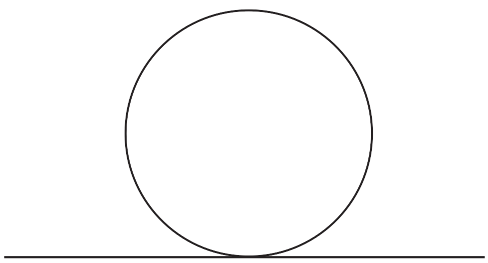
5. Add a factor $-g^2$ for each vertex and consider energy and momentum conservation at each vertex.

4.5 Thermal Mass

The thermal mass, sometimes also called self-energy, screening mass, or Debye mass, is a correction to the free propagator that comes from the continuous interaction of the particles with the surrounding medium. Thus, even massless particles acquire a mass. From now on, we set the initial mass to zero without loss of generality. All mass terms are pure corrections to the vanishing physical mass.

As mentioned earlier, a propagator shifts a particle from one point to another, so the diagrammatic figure to draw is a straight line:

The first order correction to the line is a bubble, sitting in the middle of the line (this can be derived and shown exactly, see [6]):



At first we determine the combinatoric factor: There are six different possibilities to connect two ends of the cross. Then we have to multiply the factor by two, considering that we could change the role of the in- and outgoing particle, so the combinatoric factor is twelve. Additionally we have one vertex (factor $-g^2$) and one line (factor $T \cdot \sum_n \int \frac{d^3p}{(2\pi)^3} \mathcal{D}_0(\omega_n, \mathbf{p})$, the ingoing and outgoing line can be truncated, it carries no information and is therefore not considered):

$$m_D^2 = 12g^2 T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_n^2 + \mathbf{p}^2} \,. \tag{40}$$

To evaluate the sum over the Matsubara frequencies, we use a trick instead of a formula for the sum, which is contour integration.

4.5.1 Sum over Matsubara Frequencies

In this chapter, we start with a short reminder of the residue theorem: If $f(z)$ has only isolated singularities z_n and is everywhere else analytic, we can rewrite the contour integral as a sum over the residues that are enclosed by the contour path.

$$\frac{1}{2\pi i} \oint_C dz\, f(z) = \sum_n \mathrm{Res}\, f(z) \mid_{z=z_0} \,. \tag{41}$$

If we can write the function f as fraction of two analytic functions $f(z) = \frac{\varphi(z)}{\psi(z)}$, the residue can be calculated easily:

$$\mathrm{Res} f(z) \mid_{z_o} = \frac{\varphi(z_n)}{\psi'(z_n)} \,. \tag{42}$$

In this case, we use the residue theorem the other way round: For the sum over Matsubara frequencies we search for a function with poles in such a way that the contour integral over the poles is the same as the original sum. We want to calculate a Matsubara sum of the following form

$$T \sum_{n=-\infty}^{\infty} f(p_0 = i\omega_n = 2\pi n T i) \tag{43}$$

where we have introduced p_0 as the first component of a four-vector. The following integral is the expression of Eq.(43) as a contour integral:

$$\frac{T}{2\pi i} \oint_c dp_0 f(p_0) \frac{1}{2} \beta \coth\left(\frac{1}{2}\beta p_0\right) \,. \tag{44}$$

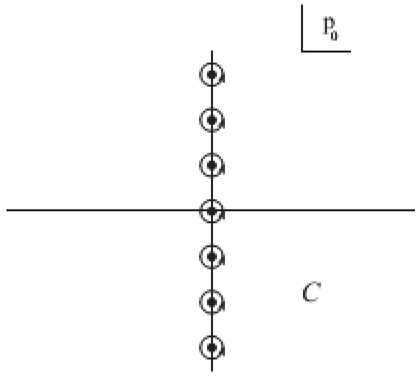
For the proof, we use Eq. (42) and $\coth(x) = \frac{\cosh(x)}{\sinh(x)}$. The poles of the $\coth(x)$ are:

$$\begin{aligned} \sinh\left(\frac{1}{2}\beta p_0\right) &= 0 \,, \\ \frac{1}{2}\beta p_0 &= \pi n i \,, \\ p_0 = 2\pi n T i &= i\omega_n \,. \end{aligned} \tag{45}$$

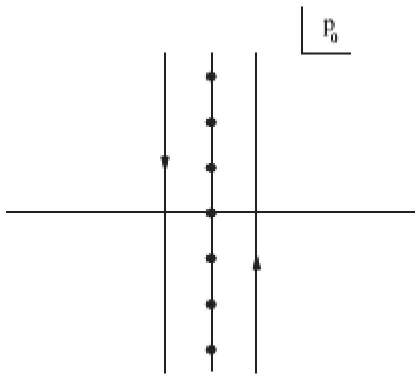
Now we can compute the residues:

$$\begin{aligned} & 2\pi i \sum \mathrm{Res}\left\{\frac{T}{2\pi i} f(p_0)\frac{1}{2}\beta\frac{\cosh\left(\frac{1}{2}\beta p_0\right)}{\sinh\left(\frac{1}{2}\beta p_0\right)}\right\} \\ = & \sum_n \frac{2\pi i}{2\pi i} T\cdot f(i\omega_n)\frac{1}{2}\beta\frac{\cosh(i\omega_n)}{\cosh(i\omega_n)}\cdot\frac{2}{\beta} \\ = & \sum_n T\cdot f(i\omega_n)\ q.e.d. \end{aligned} \tag{46}$$

The chosen contour looks like this [6]:



Beside the poles, the chosen function is everywhere analytic and bounded, therefore we can deform the contour slightly in such a way that the infinitely slim contour encloses all singularities along the imaginary axis at once [6]:



Now we have to rearrange the exponential functions of the $\coth(\frac{1}{2}\beta p_0)$ in the following way:

$$\begin{aligned} \coth\left(\frac{x}{2}\right) &= \frac{\cosh\left(\frac{x}{2}\right)}{\sinh\left(\frac{x}{2}\right)} = \frac{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}} = \frac{e^{\frac{x}{2}}}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}} + \frac{e^{-\frac{x}{2}}}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}} \\ &= \frac{1}{e^{-\frac{x}{2}}\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)} + \frac{1}{e^{\frac{x}{2}}\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)} \\ &= \frac{1}{1-e^{-x}} + \frac{1}{e^{x}-1} = -\frac{1}{2} - \frac{1}{e^{-x}-1} + \frac{1}{2} + \frac{1}{e^{x}-1}\,. \end{aligned} \tag{47}$$

This leads to the expression

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{i\infty-\epsilon}^{-i\infty-\epsilon} dp_0 f(p_0)\left(-\frac{1}{2}-\frac{1}{e^{-\beta p_0}-1}\right)+ \\
&\qquad \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} dp_0 f(p_0)\left(\frac{1}{2}+\frac{1}{e^{\beta p_0}-1}\right)
\end{aligned}
\tag{48}
$$

where we set for simplicity $p_0 \to -p_0$, change the roles of the limits of the integration to pick up a second minus sign and rearrange the functions $f(p_0)$:

$$
T\sum_{n=-\infty}^{\infty} f(p_0 = i\omega_n) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp_0 \frac{1}{2}\left[f(p_0)+f(-p_0)\right] \tag{49}
$$

$$
+ \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} dp_0 \left[f(p_0)+f(-p_0)\right]\left(\frac{1}{e^{\beta p_0}-1}\right). \tag{50}
$$

The epsilon in the integration limits simply denotes the width of the chosen contour.

4.5.2 Vacuum Part of the Screening Mass

In this equation we insert $f(p_0) = -\frac{1}{p_0^2-\mathbf{p}^2}$ to calculate Eq. (40). Note that p_0 appears only quadratic, therefore $f(p_0) = f(-p_0)$.

$$
\begin{aligned}
&\tfrac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp_0\, \tfrac{2}{2} f(p_0) + \tfrac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp_0 2f(p_0)\tfrac{1}{e^{\beta p_0}-1} \\
= &\tfrac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp_0 \tfrac{1}{-p_0^2+\mathbf{p}^2} + \tfrac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp_0 \tfrac{2}{e^{\beta p_0}-1}\tfrac{1}{-p_0^2+\mathbf{p}^2}\,.
\end{aligned}
\tag{51}
$$

To compute the first integral, we introduce a new variable $p_4 = ip_0$ as fourth component of a four-vector $p = (\mathbf{p}, p_4)$. Considering $dp_4 = i \cdot dp_0$ we obtain the vacuum part in first order after multiplying with the prefactor of the screening mass (see Eq. (40)):

$$
m_{D,vac}^2 = 12g^2\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\int_{-\infty}^{\infty} dp_4 \frac{1}{p_4^2+\mathbf{p}^2} = 12g^2\int\frac{d^4p}{(2\pi)^4}\frac{1}{p_4^2+\mathbf{p}^2}\,. \tag{52}
$$

The minus sign coming from the i^2 cancels with the minus that is acquired by changing the limits of the second integration.

This integral is independent of the temperature, therefore it exists also in the limit $T \to 0$. This is why it is called the vacuum part of the screening mass. The problem is that this expression is actually divergent. This divergence has to be regulated, for instance with a high-momentum cut-off Λ_c on $p = \sqrt{p_4^2+\mathbf{p}^2}$. Since there is no dependence on an angle, we can switch to spherical coordinates in four dimensions and integrate the first part of the Jacobian determinant, which simply yields a factor of $2\pi^2$.

$$
m_{D,vac}^2 = \frac{3g^2}{2\pi^2}\int_0^{\Lambda_c} dp\,\frac{\mathbf{p}^3}{\mathbf{p}^2} = \frac{3g^2}{4\pi^2}\Lambda_c^2 \tag{53}
$$

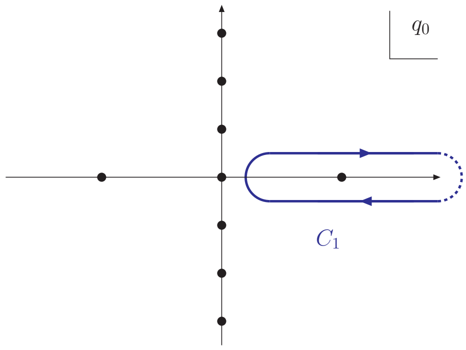
To regulate this we add a counterterm $-\frac{1}{2}\delta m^2\phi^2$ to the Lagrangian, which contributes to the screening mass simply with δm^2. We choose the counterterm in such a way that

$$m^2_{D,vac,ren} = \frac{3g^2}{4\pi^2}\Lambda_c^2 + \delta m^2 = 0\,.$$

This can be done to any order of the coupling, which is described exactly in [6].

4.5.3 Matter Part of the Screening Mass at $\mathbf{O(g^2)}$

For the second integral in Eq. (49) we deform the contour a last time so that it encloses only one pole on the real axis, namely $p_0 = +|\mathbf{p}|$ [1]:



The formula to compute the residue is quite easy: If a function $f(z)$ has a pole z_0 of order n, the residue of $f(z)$ is

$$\mathrm{Res}_{z_0} f = \frac{1}{(n-1)!}\lim_{z\to z_0}\frac{\partial^{n-1}}{\partial z^{n-1}}\left[(z-z_0)^n f(z)\right]\,. \tag{54}$$

Following the same procedure as before, we compute for the matter part of the screening mass:

$$\begin{aligned} m^2_{D,mat} &= 12g^2\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp_0\frac{2}{e^{\beta p_0}-1}\cdot\frac{1}{-p_0^2+\mathbf{p}^2} \\ &= -12g^2\int\frac{d^3p}{(2\pi)^3}\frac{2\pi i}{2\pi i}\lim_{p_0\to|\mathbf{p}|}\frac{2}{e^{\beta p_0}-1}\frac{(-p_0+|\mathbf{p}|)}{(|\mathbf{p}|-p_0)(|\mathbf{p}|+p_0)} \\ &= 12g^2\int\frac{d^3p}{(2\pi)^3}\frac{1}{|\mathbf{p}|}\frac{1}{e^{\beta|\mathbf{p}|}-1} = \frac{6g^2}{\pi^2}\int_0^\infty dp\,|\mathbf{p}|\,n\left(|\mathbf{p}|\right)\,. \end{aligned} \tag{55}$$

In this calculation we pick up an additional minus sign from the second to the third line: It comes in by the integration in clockwise direction, that means in the mathematical negative way. $n(p)$ is the Bose-Einstein distribution function:

$$n(q) = \frac{1}{e^{q/T}-1}\,. \tag{56}$$

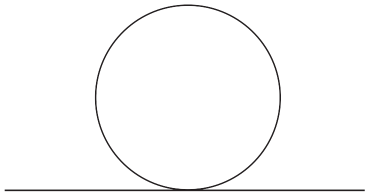
This integration can be calculated analytically and yields the important result:

$$m^2_{D,mat} = g^2T^2\,. \tag{57}$$

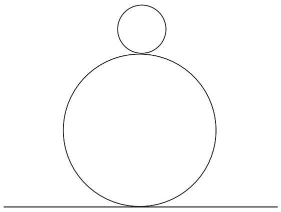
To sum up, we separated the thermal mass in two parts, the matter and the vacuum part and renormalized the first part to zero. The second part vanishes at $T = 0$ and is also momentum independent, but this is not necessarily true in every power of the coupling.

4.5.4 Screening Mass at Next To Leading Order

The first idea to compute the thermal mass is to draw two bubbles on a line next to each other.



Although we have two vertices in this diagram, it does not contribute to the thermal mass at the next to leading order. The reason for that is the fact that the diagram is not 1 PI (one particle irreducible). This means that we could separate the two bubbles by cutting just one line and so the diagram is not connected. To avoid this, we draw a second loop on top of our first loop:



This diagram consists of one line from the first vertex to the second with momentum p, one closed loop with momentum q and of a third line from the second vertex back to the first one, which carries also p due to momentum conservation. The combinatoric factor for each vertex is, as mentioned earlier, 12, and each vertex comes in with a factor of $-g^2$.

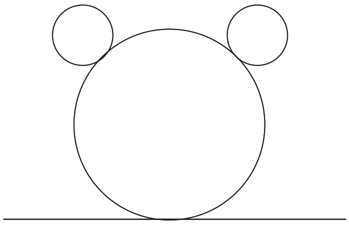
$$m_D^2 \mid_{g^4} \quad = \quad 12g^2T\sum_{\omega_n}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\omega_n + q^2} \cdot 12g^2T\sum_{\omega_m}\int \frac{d^3p}{(2\pi)^3}\frac{-1}{(\omega_n + p^2)^2} \tag{58}$$

To understand this formula we have to explain the diagrammatic rules in more detail: Each closed loop leads to an integration and the sum over the Matsubara frequencies times the temperature. Every internal line from one vertex to another comes with its own propagator, this is why the second propagator is squared.

The first part is just m^2 at $O(g^2)$. Looking at the second integral one can easily see that for $\omega_m = 0$ the integral is IR (infrared) divergent (that means in the language of theoretical physicists for small momenta and not in the strict way.)

$$\omega_m = 0 : \int_0^\infty \frac{dp}{p^2}\,\text{IR divergent}$$

At $O(g^6)$ the first integral gives m^4, in the second part the divergence is even worse, what can be seen looking at the "Mickey Mouse diagram":



$$\underbrace{\left(12g^2T\sum_{\omega_n}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\omega_n+q^2}\right)^2}_{m_D^4} \cdot \underbrace{12g^2T\sum_{\omega_m}\int\frac{d^3p}{(2\pi)^3}\frac{-1}{(\omega_n+p^2)^3}}_{\omega_m=0:\ \int_0^\infty \frac{d^3p}{p^6}\ IR\,divergent}. \tag{59}$$

The divergence becomes worse and worse from order to order, while the first part always gives the screening mass $m_D^{N-2}\,|_{g^2}$ to the power of $N-2$, where N refers to the current order. To solve the IR-divergences we sum up all the IR divergent diagrams in the following way:

$$\begin{aligned} G_0 + m_D^2G_0^2 + m_D^4G_0^3 + m_D^6G_0^4 + \ldots &= \sum_{n=1}^{\infty} G_0^n m_D^{2n-2} = \frac{1}{m_D^2}\sum_{n=1}^{\infty} G_0^n m_D^{2n} \\ = \frac{1}{m_D^2}\sum_{n=1}^{\infty}\left(G_0m_D^2\right)^n = \frac{1}{m_D^2}\left(\sum_{n=0}^{\infty}\left(G_0m_D^2\right)^n - 1\right) &= \frac{1}{m_D^2}\cdot\left(\frac{1}{1-m_D^2G_0}-1\right) \\ &= \frac{-1}{-G_0^{-1}+m_D^2} = \underbrace{\frac{-1}{\omega_n^2+p^2+m_D^2}}_{G}. \end{aligned} \tag{60}$$

In the second line we use the limit of the geometric sum $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. In contrast to the bare propagator G_0 we call G the dressed propagator. In this formula we can see why we call the screening mass a correction to the free propagator. In the first line we add G_0 artificially in order to build the sum correctly, so we have to subtract it again under the integral. This is identical with subtracting the $O(g^2)$ contribution to obtain a clean $O(g^3)$ part. The equation which we have to solve now is the following:

$$m_D^2\,|_{g^3} \quad = \quad 12g^2T\sum_{\omega_n}\int\frac{d^3q}{(2\pi)^3}\left(\frac{1}{\omega_n^2+q^2+m_D^2}-\frac{1}{\omega_n^2+q^2}\right). \tag{61}$$

Only the first mode, $n = 0$, contributes to the next to leading order. Remembering $\omega_n = 2\pi nT$, we set $\omega_n = 0$ and calculate the integral analytically:

$$\begin{aligned} m_D^2 \mid_{g^3} &= 12g^2T\sum_{\omega_n}\int \frac{d^3q}{(2\pi)^3}\left(\frac{1}{q^2+m_D^2}-\frac{1}{q^2}\right) \\ &= 12g^2T\int \frac{4\pi q^2\, dq}{(2\pi)^3}\frac{(-m_D^2)}{q^2(q^2+m_D^2)} \\ &= \frac{6g^2}{\pi^2}T\int_0^\infty dq\, \frac{-m_D^2}{q^2+m_D^2} \\ &= -\frac{3}{\pi}g^2Tm_D \\ &= -\frac{3}{\pi}g^3T^2\,. \end{aligned} \tag{62}$$

In the last line we inserted $m_D \mid_{g^2}= gT$.

The calculation of the screening mass at higher order of perturbation theory is beyond the scope of this thesis, so the further results are taken from literature [2]:

$$m_D^2 = g^2T^2\left\{1-\frac{3}{\pi}g-\frac{9g^2}{2\pi^2}\left[\ln\frac{\mu}{2\pi T}-\frac{4}{3}\ln\frac{g}{\pi}-2.415\right]\right\}+O(g^5)\,. \tag{63}$$

4.6 Pressure

The lowest order correction to the thermal pressure is given by Fig. 1. Using the Feynman rules and our knowledge of Matsubara sums we can easily compute $\ln(Z_1)$:

$$\ln(Z_1) = -3g^2\left(T\sum_n\int\frac{d^3p}{(2\pi)^3}\mathcal{D}_0(\omega_n,\mathbf{p})\right)^2 \tag{64}$$

$$\begin{aligned} &= -3g^2\left(\int\frac{4\pi p^2dp}{(2\pi)^3}\frac{1}{p}n(p)\right)^2 = \frac{-3g^2}{2\pi^2}\left(\int dp\,p\cdot n(p)\right)^2 \\ &= \frac{-3g^2}{4\pi^4}\left(\frac{T^2\pi^2}{6}\right)^2 \\ &= -\frac{g^2T^2}{48}\,. \end{aligned} \tag{65}$$

Actually we also have to mention the $-\delta m^2\phi^2$ term of the Lagrangian. This part is a divergent contribution to the vacuum pressure at $T = 0$ and the zero-point energy. Since the absolute pressure is not measurable (we can only measure energy and pressure differences) we can neglect the divergent part after we renormalize the vacuum pressure to zero.

For the next order, the same problems arise as before, the integrals are IR divergent, so we apply the resummation scheme once again to compute the next to leading order correction for the pressure. At the end, we obtain for the pressure [2]:

$$P = P_0\left[1-\frac{15}{8}\left(\frac{g}{\pi}\right)^2+\frac{15}{2}\left(\frac{g}{\pi}\right)^3+O(g^4)\right]\,. \tag{66}$$

$P_0 = T^4 \left(\frac{\pi^2}{90}\right)$ is the pressure of an ideal gas for massless bosons. It can be computed directly from the path integral with the original scalar Lagrangian without the interaction term, see for instance [7] or [6].

4.7 Running of the Coupling

The coupling is, as mentioned earlier, not a constant. The running of the coupling is gained by the solution of the one-loop β-function.

$$\beta(g^2) \equiv \mu \frac{\partial g^2}{\partial \mu} = \frac{9}{2\pi^2} g^4 + O(g^6)\,. \tag{67}$$

This differential equation can be solved using the separation of variables:

$$\frac{1}{g^4} dg^2 = \frac{9}{2\pi^2} \frac{1}{\mu} d\mu\,. \tag{68}$$

Introducing the ultraviolet cut-off Λ and the value of the coupling at Λ, g_Λ, we integrate both sides and obtain:

$$\frac{1}{g_\mu^2} = \frac{1}{g_\Lambda^2} + \frac{9}{2\pi^2} \ln \frac{\Lambda}{\mu}\,. \tag{69}$$

For $\mu \to 0$, we obviously get $g_\mu \to 0$, therefore the coupling at $\mu = 0$ is not a good quantity to characterize the theory, so we have chosen the usual practice to fix the coupling at $\mu = 2\pi T$ and denote $g(\mu = 2\pi T)$ simply as g.

The second thing to mention is that the coupling already becomes infinite at a finite value of $\mu = \Lambda_L$:

$$\Lambda_L = 2\pi T \cdot e^{\frac{2\pi^2}{9g^2}}\,.$$

This is called a Landau pole, which means that in contrast to QCD, ϕ^4-theory is not asymptotically free. It is possible that the pole is just a relict of the perturbation computation of the β-function, i.e. a sign that the perturbative approximation breaks down at strong coupling.

5 Non-Perturbative Renormalization Group

Unless the coupling is very small, $g \lesssim 0.1$, the corrections at each order of perturbation theory for the thermal mass and the pressure do not decrease in magnitude. This is in contrast to what we would expect for a good perturbative expansion where we want to neglect terms of higher order. This is why we search for a method to consider the whole path integral. One possible way to do this is the non-perturbative renormalization group (NPRG) formalism.

The basic idea of the NPRG is to add a regulator ΔS_κ to the classical action S. This regulator comes with a continuous parameter κ and enters into the Lagrangian with ϕ^2, so it plays the role of an additional mass term. ΔS_κ should totally suppress the quantum fluctuations. To be more exact, the strategy is as follows: We start with given physical parameters at $\kappa = 0$ and $T = 0$, turn on κ and compute the bare quantities at $T = 0$ and then follow the flow back to $\kappa = 0$ now with $T \neq 0$. So we are able to obtain the physical quantities in our full quantum theory, concerning all fluctuations. [1]

5.1 Derivation of the Flow Equation

To make this work with have to look how the regulator changes if we modify κ. This means that we have to derive a flow equation for the effective potential $\Gamma_\kappa [\langle\phi\rangle]$. Therefore we start with the regulator in momentum space, i.e. the Fourier transform of ΔS_κ,

$$\Delta S_\kappa = \frac{T}{2} \sum_{\omega_n} \int \frac{d^{d-1}\mathbf{q}}{(2\pi)^{d-1}} \varphi(q) R_\kappa \varphi(-q) \,, \tag{70}$$

where q is defined in four-dimensional momentum space as before: $q = (q_0, \mathbf{q})$. $q_0 = i\omega_n$ are the Matsubara frequencies mentioned earlier. R_κ is a cut-off function which should go to zero for high momenta and to κ^2 for $q < \kappa$. Since R_κ is given in momentum space, we have to perform a Fourier transformation of our fields, which is denoted by $\varphi(q)$. The connected Green's function or, speaking in words of statistical mechanics, the Helmholtz free energy is given as the natural logarithm of the partition function Z_κ. In the path-integral formalism we can formally compute all physical quantities by adding a source term to the Lagrangian. The free energy under the influence of the source j now becomes

$$W_\kappa[j] = \ln\left(\int \mathcal{D}\varphi \, e^{-S[\varphi] - \Delta S_\kappa[\varphi] + \int d^4x \, j\varphi} \right) = \ln(Z_\kappa) \,. \tag{71}$$

In the further derivation of the flow equation, we will introduce a shorthand notion for the Integral over the spatial components of q and the sum over the Matsubara frequencies: $T \cdot \sum_{\omega_n} \int \frac{d^{d-1}\mathbf{q}}{(2\pi)^{d-1}} \to \int_q$ and $\varphi(q) \to \varphi_q$. The other integrals correspond to the three-dimensional integral over space to switch from density quantities to full quantities, for example the current density j.

Due to our introduction of the source term j in our functional $W_\kappa[j]$, we can obtain the expactation value of our field φ by functional derivation of W with respect to j:

$$\phi_{\kappa,J} = \langle\phi\rangle_{\kappa,J} = \frac{\delta W_\kappa}{\delta J}\,, \tag{72}$$

$$\frac{\delta W_\kappa}{\delta J} = \frac{\delta}{\delta J}\ln(Z_\kappa) = \frac{1}{Z_\kappa}\frac{\delta}{\delta J}\int \mathcal{D}\varphi\, e^{-S[\varphi]-\Delta S_\kappa[\varphi]+\int d^4x\, j\varphi} =$$

$$= \frac{1}{Z_\kappa}\int \mathcal{D}\varphi\,\varphi(x) e^{-S[\varphi]-\Delta S_\kappa[\varphi]+\int d^4x\, j\varphi} = \phi_{\kappa,J}\,. \tag{73}$$

Here we recognize the expectation value of the field, see Eq. (24), so we proved Eq. (72). To gain our effective potential $\Gamma_\kappa[\langle\phi\rangle]$, we have to perform a Legendre transformation.

$$\Gamma_\kappa[\langle\phi\rangle] = -W_\kappa[j] + \int_q j\phi - \Delta S_\kappa \tag{74}$$

A bare Legendre transformation would not include the last term, it has to be added by hand. For $\kappa \to 0$ we have $W_\kappa \to 0$ and $\Delta S_\kappa \to 0$ and therefore $\Gamma_\kappa \to \Gamma$. We see that the additional term does not spoil the $\kappa \to 0$ limit. At the other end of the scale, $\kappa \to \Lambda$, we need this term to restore our full quantities. For an exact derivation see [4].

Since we want to compute the change of Γ with respect to κ, we build the partial derivative of the effective potential with respect to our scale parameter:

$$\frac{\partial \Gamma_\kappa}{\partial\kappa} = \frac{-\partial W_\kappa}{\partial\kappa} + \frac{\partial}{\partial\kappa}\int_q (j\phi) - \frac{1}{2}\int_q \phi_q \partial_\kappa R_\kappa \phi_{-q}\,, \tag{75}$$

$$\frac{\partial W_\kappa}{\partial\kappa} = \frac{1}{Z_\kappa}\int \mathcal{D}\varphi\, e^{-S[\varphi]-\Delta S_\kappa[\varphi]+\int d^4x\, j\varphi}\left\{-\frac{1}{2}\int_q \varphi_q\partial_\kappa R_\kappa \varphi_{-q} + \int_q \partial_\kappa j_\kappa \varphi\right\}. \tag{76}$$

Using our definition of an expectation value, the path integral vanishes by replacing all φ by $\phi = \langle\varphi\rangle$:

$$\frac{\partial W_\kappa}{\partial\kappa} = -\frac{1}{2}\int_q \langle\varphi_q\varphi_{-q}\rangle\,\partial_\kappa R_\kappa(q) + \int_q \partial_\kappa\left(j_\kappa\phi\right). \tag{77}$$

In the second term the integral is just the three-dimensional integral, arising because of the current density j_κ. If we insert this in Eq. (75), the term proportional to j drops out and we obtain our first result:

$$\frac{\partial\Gamma_\kappa}{\partial\kappa} = \frac{1}{2}\int_q \partial_\kappa R_\kappa(q)\left(\langle\varphi_q\varphi_{-q}\rangle - \phi_q\phi_{-q}\right). \tag{78}$$

This result can be simplified after taking a look at the second variation of Γ with respect to the field:

$$\frac{\delta\Gamma}{\delta\phi} = -\int \underbrace{\frac{\delta W}{\delta j}}_{\phi}\frac{\delta j}{\delta\phi} + j + \int \phi\frac{\delta j}{\delta\phi} - \int_q \phi_q R_\kappa(q) = j - \int_q \phi_q R_\kappa(q)\,. \tag{79}$$

This derivation is not very strict in a mathematical sense. Various δ-functions arise and vanish after integration in this line. Now we vary Γ a second time:

$$\frac{\delta^2\Gamma}{\delta\phi\delta\phi} = \frac{\delta j}{\delta\phi} - R_\kappa(q)\,. \tag{80}$$

Since $\frac{\delta W}{\delta j} = \phi$, it is clear that $\frac{\delta^2 W}{\delta j^2} = \frac{\delta\phi}{\delta j}$ and therefore $\frac{\delta j}{\delta\phi} = \left(\frac{\delta^2 W}{\delta j^2}\right)^{-1}$, so we have to calculate this expression:

$$\begin{aligned}
\frac{\delta W}{\delta J} &= \frac{1}{Z_\kappa}\int \mathcal{D}\varphi\,\varphi e^{-S(\varphi)-\Delta S(\varphi)+\int j\varphi} = \phi\,, \\
\frac{\delta^2 W}{\delta J^2} &= -\frac{1}{Z_\kappa^2}\left(\int \mathcal{D}\varphi\,\varphi e^{-S(\varphi)-\Delta S(\varphi)+\int j\varphi}\right)^2 + \frac{1}{Z_\kappa}\int \mathcal{D}\varphi\,\varphi\varphi e^{-S(\varphi)-\Delta S(\varphi)+\int j\varphi}\,, \\
\frac{\delta^2 W}{\delta J^2} &= \langle\varphi\varphi\rangle - \langle\varphi\rangle\langle\varphi\rangle\,.
\end{aligned} \tag{81}$$

This shows that $(\langle\varphi\varphi\rangle - \langle\varphi\rangle\langle\varphi\rangle)^{-1} = \frac{\delta^2\Gamma}{\delta\phi\delta\phi} + R_\kappa$ what yields our final result for the flow equation:

$$\partial_\kappa\Gamma_\kappa[\phi] = \frac{T}{2}\sum_{\omega_n}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\partial_\kappa R_\kappa(\mathbf{q})\left[\Gamma_\kappa^{(2)} + R_\kappa\right]^{-1}\,. \tag{82}$$

5.2 Local Potential Approximation

The local potential approximation is a method to solve the flow equation at zero external momentum. We assume that Γ_κ has a simple form:

$$\Gamma_\kappa^{LPA}[\phi] = \int_0^\beta d\tau\int d^3x\left\{\frac{1}{2}\partial^\mu\phi\partial_\mu\phi + V_\kappa(\rho)\right\}\,. \tag{83}$$

In this equation, $V_\kappa(\rho)$ denotes the effective potential and ρ simply is an abbreviation for $\rho = \frac{\phi^2}{2}$. If we insert this ansatz into both sides of Eq. (82), we obtain a new flow equation for the effective potential. On the l.h.s., we build the derivation with respect to κ. Since only the effective potential depends on κ, all the other terms vanish. On the r.h.s., the second functional derivative of Γ_κ^{LPA} with respect to ϕ occurs. At first, we only take a look at the part containing $\partial^\mu\phi\partial_\mu\phi$. To compute the behavior of the derivatives under functional integration, we apply once again a Fourier transformation to the field.

$$\phi(x) = \int d^4q\, e^{iqx}\phi(q) \tag{84}$$

Then we form the first functional derivative:

$$\begin{aligned}\frac{\delta}{\delta\phi}\left(\partial^\mu\phi\partial_\mu\phi\right) &= \frac{\delta}{\delta\phi(q')}\int d^4q\,\partial^\mu e^{iq\cdot x}\phi(q)\int d^4p\,\partial_\mu e^{ip\cdot x}\phi(-p) \\ &= \frac{\delta}{\delta\phi(q')}\int d^4pd^4q\,(i^2)q\cdot p\,e^{i(q+p)\cdot x}\phi(q)\phi(-p) \\ &= -\int d^4pd^4q\,q\cdot p\,\delta(q-q')e^{i(q+p)\cdot x}\phi(-p) \\ &\quad -\int d^4p\,d^4q\,q\cdot p\,\delta(p+q')e^{i(q+p)\cdot x}\phi(q) \\ &= -\int d^4p\,q'\cdot p\,e^{i(q'+p)\cdot x}\phi(-p) \\ &\quad +\int d^4q\,q\cdot q'e^{i(q-q')\cdot x}\phi(q)\,. \end{aligned} \tag{85}$$

In the second line, the index structure is hidden in the Minkowski scalar product $p\cdot q$. Now we calculate the second derivative:

$$\begin{aligned}\frac{\delta^2}{\delta\phi^2}\left(\partial^\mu\phi\partial_\mu\phi\right) &= \frac{\delta}{\delta\phi(q')}\Big\{-\int d^4p\,q'\cdot p\,e^{i(q'+p)\cdot x}\phi(-p) \\ &\quad +\int d^4q\,q\cdot q'e^{i(q-q')\cdot x}\phi(q)\Big\} \\ &= -\int d^4p\,q'\cdot p\,e^{i(q'+p)\cdot x}\delta(p+q') \\ &\quad +\int d^4q\,q\cdot q'e^{i(q-q')\cdot x}\delta(q-q') \\ &= q'\cdot q'e^{(q'-q')\cdot x}+q'\cdot q'e^{i(q'-q')\cdot x}=q'^2e^0+q'^2e^0 \\ &= 2q^2\,. \end{aligned} \tag{86}$$

The second part is easier, we just have to vary the potential with respect to ϕ, where $V'(\rho)$ denotes the derivative with respect to the argument of the potential, i.e. ρ.

$$\begin{aligned}\frac{\delta}{\delta\phi}\left(\frac{\delta V(\rho)}{\delta\phi}\right) &= \frac{\delta}{\delta\phi}\left(\frac{dV}{d\rho}\cdot\frac{d\rho}{d\phi}\right)=\frac{\delta}{\delta\phi}\left(V'(\rho)\phi\right)=V''(\rho)\cdot\phi\phi+V'(\rho) \\ &= V''(\rho)2\rho+V'(\rho)\,. \end{aligned} \tag{87}$$

After inserting this results into the original flow equation, we obtain a new flow equation for the effective potential:

$$\partial_\kappa V_\kappa(\rho)=\frac{1}{2}T\sum_{\omega_n}\int\frac{d^3q}{(2\pi)^3}\left[\partial_\kappa R_\kappa(q)\right]\frac{1}{q^2+V'(\rho)+2\rho V''(\rho)+R_\kappa(q)}\,. \tag{88}$$

A convenient choice for the regulator, which only depends on the spatial components of q and fulfills the constraints given in the introduction of this chapter, is

$$R_\kappa(\mathbf{q})=\left(\kappa^2-\mathbf{q}^2\right)\theta(\kappa^2-\mathbf{q}^2)\,. \tag{89}$$

Inserting this in the latter equation and remembering $q^2 = \mathbf{q}^2 + \omega_n^2$, we obtain our final result for the local potential approximation:

$$\begin{aligned}
\partial_\kappa V_\kappa(\rho) &= \frac{1}{2} T \sum_{\omega_n} \int_0^\kappa \frac{d^3q}{(2\pi)^3} \frac{\left[\partial_\kappa \left(\kappa^2 - \mathbf{q}^2\right) \theta(\kappa^2 - \mathbf{q}^2)\right]}{\omega_n^2 + \mathbf{q}^2 + V'(\rho) + 2\rho V''(\rho) + \kappa^2 - \mathbf{q}^2} \\
&= \frac{1}{2} T \sum_{\omega_n} \int_0^\kappa \frac{dq}{(2\pi)^3} 4\pi \mathbf{q}^2 \frac{2\kappa}{\omega_n^2 + V'(\rho) + 2\rho V''(\rho) + \kappa^2} \\
&= \frac{T}{6\pi^2} \mathbf{q}^3 \Big|_0^\kappa \sum_{\omega_n} \frac{\kappa}{\omega_n^2 + V'(\rho) + 2\rho V''(\rho) + \kappa^2} \\
&= \frac{T}{6\pi^2} \sum_{\omega_n} \frac{\kappa^4}{\omega_n^2 + V'(\rho) + 2\rho V''(\rho) + \kappa^2} \,.
\end{aligned} \tag{90}$$

In the first line, the Heaviside function in the denominator is covered by the new limits of the integration. In the second line we would obtain a second term because of the derivation of the second theta function, but this term is proportional to $\left(\kappa^2 - \mathbf{q}^2\right) \delta \left(\kappa^2 - \mathbf{q}^2\right)$ and therefore zero. To perform the Matsubara sum we introduce a formula for this special case, which can be obtained directly using the methods introduced in chapter 3:

$$T \sum_{\omega_n} \frac{1}{-(i\omega_n)^2 + \omega_\kappa^2} = \frac{1 + 2n(\omega_\kappa)}{2\omega_\kappa} \,. \tag{91}$$

Setting $\omega_\kappa^2 = V'(\rho) + 2\rho V''(\rho) + \kappa^2$, we finally derive

$$\partial_\kappa V_\kappa(\rho) = \frac{\kappa^4}{6\pi^2} \left\{ \frac{2n(\omega_\kappa) + 1}{2\omega_\kappa} \right\} \,. \tag{92}$$

This equation is derived without using any approximations or perturbative expansions.

6 Perturbative Analysis of the NPRG

In principle, it is possible to integrate the flow equation for the effective potential numerically, for instance using a Runge-Kutta method. Another possibility is to extract the results from perturbation theory order by order. We will show, that up to order g^3 the results will be completely the same and start to differ from order g^4 because of different renormalization schemes. In this thesis, we will concentrate on the thermal mass and the coupling as far as it is needed to compute the mass.

6.1 Truncation of the Flow Equations

Let us consider a simplified version of Eq. (92) where we expand the potential $V(\rho)$ around $\rho = 0$. The potential is constructed in such a way, that $m_\kappa = V_\kappa^{'}(\rho)\mid_{\rho=0}$ and $g_\kappa^2 = \frac{V_\kappa^{''}(\rho)|_{\rho=0}}{8}$:

$$\begin{aligned} V_\kappa(\phi) &= \mathcal{V}_\kappa + \frac{m_\kappa^2}{2}\phi^2 + g_\kappa^2\phi^4 + h_\kappa^2\phi^6 \\ &= \mathcal{V}_\kappa + m_\kappa^2\rho + 4g_\kappa^2\rho^2 + 8h_\kappa^2\rho^3\,. \end{aligned} \tag{93}$$

In this equation m_κ represents the thermal mass under the influence of the scale parameter κ and so on. With this definition also ω_κ changes:

$$\omega_\kappa \;=\; V'(\rho) + 2\rho V''(\rho) + \kappa^2 \quad = \quad \kappa^2 + m_\kappa^2 + 24 g_\kappa^2\rho\,. \tag{94}$$

The next step is to truncate the potential at order $O(\rho^3)$ which means to neglect h_κ and insert it into the l.h.s. of Eq. (92). Then we are able to expand also the r.h.s. and compare them order by order.

In order to do the expansion of the r.h.s. we first define ϵ_κ as ω_κ evaluated at $\rho = 0$:

$$\epsilon_\kappa^2 = m_\kappa^2 + \kappa^2\,. \tag{95}$$

Since in the following derivation the derivative of the distribution function with respect to ρ will appear, I want to mention that $\frac{\partial\omega_\kappa}{\partial\rho}\mid_{\rho=0}$ is proportional to the coupling g_κ^2 divided by ϵ_κ:

$$\left.\frac{\partial\omega_\kappa}{\partial\rho}\right|_{\rho=0} = \frac{1}{2}\left[V^{''}(\rho) + 2\rho V^{'''}(\rho) + 2V^{''}(\rho)\right]_{\rho=0}\cdot\left[k^2+m_\kappa^2\right]^{-\frac{1}{2}} = \frac{3V^{''}(0)}{2\epsilon_\kappa} = \frac{24g_\kappa^2}{2\epsilon_\kappa}\,. \tag{96}$$

Now we do the expansion and insert the results obtained above:

$$\begin{aligned} \frac{\kappa^4}{6\pi^2}\frac{1+2n(\omega_\kappa)}{2\omega_\kappa} \quad &= \quad \frac{\kappa^4}{6\pi^2}\frac{1+2n(\epsilon_\kappa)}{2\epsilon_\kappa} - \rho\frac{\kappa^4}{6\pi^2}\frac{1+2n(\epsilon_\kappa)-2\epsilon_\kappa n'(\epsilon_\kappa)}{\epsilon_\kappa^3} \\ &+ \quad \frac{4\cdot 9g_\kappa^4\kappa^4}{2\pi^2}\frac{1+2n(\epsilon_\kappa)-2\epsilon_\kappa n'(\epsilon_\kappa)+\frac{2}{3}\epsilon_\kappa^2 n''(\epsilon_\kappa)}{\epsilon_\kappa^5}\,. \end{aligned} \tag{97}$$

Comparison order by order with the truncated potential of Eq. (93) yields a set of coupled

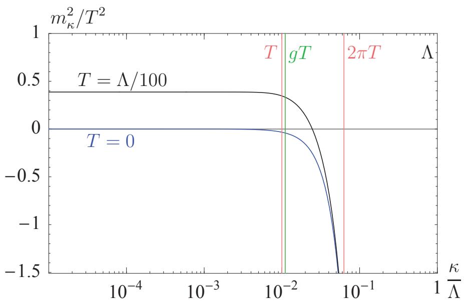


Figure 2: Flow of the mass for $g = 1.1032$. The mass converges to $m = 0.622T$ for $\kappa \to 0$. For $T = 0$, the quantity plotted is $m^2_{vac,\kappa}/T^2$ [1].

differential equations for the pressure, the thermal mass and the coupling:

$$\partial_\kappa \mathcal{V}_\kappa = \frac{\kappa^4}{6\pi^2}\frac{1+2n(\epsilon_\kappa)}{2\epsilon_\kappa}, \tag{98}$$

$$\partial_\kappa m_\kappa^2 = -\frac{g_\kappa^2\kappa^4}{\pi^2}\frac{1+2n(\epsilon_\kappa)-2\epsilon_\kappa n'(\epsilon_\kappa)}{\epsilon_\kappa^3}, \tag{99}$$

$$\partial_\kappa g_\kappa^2 = \frac{9g_\kappa^4\kappa^4}{2\pi^2}\frac{1+2n(\epsilon_\kappa)-2\epsilon_\kappa n'(\epsilon_\kappa)+\frac{2}{3}\epsilon_\kappa^2 n''(\epsilon_\kappa)}{\epsilon_\kappa^5}. \tag{100}$$

6.2 Expansion of the Flow Equation

In order to extract the results of perturbation theory presented in chapter 3, we introduce an intermediate energy scale ξ, which lies between soft and hard momenta, i.e.

$$gT \ll \xi \ll T. \tag{101}$$

In these two regions, different expansions are possible: For the hard momenta, $\kappa > \xi$, we expand ϵ_κ^2 :

$$\epsilon_\kappa^2 = \kappa^2 + m_\kappa^2 = \kappa^2\left(1+\frac{m_\kappa^2}{\kappa^2}\right). \tag{102}$$

From perturbation theory, we expect the thermal mass to be $m_{th,0} = gT$ for $\kappa = 0$. For $\kappa > \xi \gg gT$ the second term is small and we simply set

$$\epsilon_\kappa^2 = \kappa^2\left(1+O\left((gT/\xi)^2\right)\right). \tag{103}$$

This is also valid for large κ, since the thermal mass is proportional to $m_\kappa^2 = g_\kappa^2\kappa^2$ and therefore much smaller than κ, as we will see later. In the other regime, κ is small, so we are able to expand the distribution function and its derivatives:

$$n(\kappa) = \frac{T}{\kappa} - \frac{1}{2} + \frac{\kappa}{12T} + \frac{T}{\kappa}O\left((\xi/T)^3\right). \tag{104}$$

The qualitative behavior of the mass, which is useful to know for some approximations, is shown in Fig. 2. This numerical calculation is taken from literature [1]. It is to mention, that the thermal mass an the vacuum flow of the mass show the same behavior for big values of κ.

6.3 Zero-Temperature Flow

In the limit $T = 0$ all the distribution functions and its derivatives vanish because of the infinite exponential function $e^{\frac{\omega}{T}}$ in the denominator. The new set of equations can be expanded in g and solved easily:

$$\partial_\kappa \mathcal{V}_\kappa = \frac{\kappa^3}{6\pi^2} - \frac{\kappa m_\kappa^2}{12\pi^2} + \dots , \tag{105}$$

$$\partial_\kappa m_\kappa^2 = -\frac{g_\kappa^2}{\pi^2}\kappa + \dots , \tag{106}$$

$$\partial_\kappa g_\kappa^2 = \frac{9g_\kappa^4}{2\pi^2}\frac{1}{\kappa} + \dots . \tag{107}$$

For the pressure we just integrate both sides from κ' to our cut-off Λ:

$$\begin{aligned} \int_{\kappa'}^{\Lambda} d\kappa\, \partial_\kappa \mathcal{V}_\kappa &= \int_{\kappa'}^{\Lambda} d\kappa \frac{\kappa^3}{6\pi^2} , \\ \mathcal{V}_\kappa &= \mathcal{V}_\Lambda + \frac{\kappa^4 - \Lambda^4}{48\pi^2} . \end{aligned} \tag{108}$$

Next, we solve the equation for g_κ^2 by separation of the variable

$$\begin{aligned} \frac{dg_\kappa^2}{dk} &= \frac{9g_\kappa^4}{2\pi^2}\frac{1}{\kappa} , \\ \int_{\kappa'}^{\Lambda} \frac{dg_\kappa^2}{g_\kappa^4} &= \int_{\kappa'}^{\Lambda} dk \frac{9}{2\pi^2}\frac{1}{\kappa} , \\ \frac{1}{g_\Lambda^2} - \frac{1}{g_\kappa^2} &= \frac{9}{2\pi^2}\ln\left(\frac{\Lambda}{\kappa}\right) , \\ g_\kappa^2 &= \frac{g_\Lambda^2}{1 - 9g_\Lambda^2 \ln\left(\frac{\kappa}{\Lambda}\right)/(2\pi^2)} + O\left(g_\Lambda^6\right) . \end{aligned} \tag{109}$$

This equation has the structure of $\frac{x^2}{1-x^2}$. Thus, we can easily expand it in g using the geometric sum. So we can see that the first term is just g^2, which allows us to solve the equation of the thermal mass in the vacuum case simply by integration:

$$m_\Lambda^2 - m_\kappa^2 = -\frac{g^2}{2\pi^2}\left(\Lambda^2 - \kappa^2\right) . \tag{110}$$

If we assume that m_κ^2 completely arises due to quantum fluctuations, we have to adapt our integration constant m_Λ^2 in such a way that m_κ^2 vanishes for $\kappa = 0$, that means:

$$m_\Lambda^2 = -\frac{g_\Lambda^2 \Lambda^2}{2\pi^2} . \tag{111}$$

This is an important step because this new renormalization scheme leads to the difference of the results of perturbation theory and the NPRG-formalism in order g^4 what will be shown later in this thesis.

Finally, we obtain for the vacuum flow of the thermal mass the following expression:

$$m_\kappa^2 = -g_\kappa^2 \frac{\kappa^2}{2\pi^2} + O\left(g_\Lambda^4\right) . \tag{112}$$

6.4 Coupling at Finite Temperature

The main problem for solving the flow equation is that the coupling as well as the thermal mass appear in both differential equations. Thus, they are coupled. To solve this problem, we start with the unexpanded equation for the coupling Eq. (100) and write the thermal mass, appearing in $\epsilon_\kappa^2 = m_\kappa^2 + \kappa^2$, as the leading order contribution m^2 and a correction δm_κ^2: $m_\kappa^2 = m^2 + \delta m_\kappa^2$. This is not necessary for the regime $\kappa \gtrsim \xi$, because thermal effects are subleading there. The solution here can be obtained directly by expanding the vacuum solution Eq. (109) in g (remember $g_\Lambda(\Lambda = 2\pi T) = g$):

$$g_\kappa^2 = g^2 + \frac{9g^4}{2\pi^2} \ln \frac{\kappa}{2\pi T} + O\left(g^6\right) . \tag{113}$$

In the region $\kappa \lesssim \xi$ we can expand the distribution functions and their derivatives:

$$1 + 2n(\epsilon_\kappa) - 2\epsilon_\kappa n'(\epsilon_\kappa) + \frac{2}{3}\epsilon_\kappa^2 n''(\epsilon_\kappa) = \frac{16T}{3\epsilon_\kappa} + \frac{\epsilon_\kappa^5}{5670T^5} + O\left(\frac{\epsilon_\kappa^6}{T^6}\right) . \tag{114}$$

For the leading contribution it is sufficient to keep the first term, so the flow equation simply becomes

$$\partial_\kappa g_\kappa^2 = \frac{24g_\kappa^4}{\pi^2} \frac{\kappa^4 T}{\epsilon_\kappa^6} . \tag{115}$$

Now we insert our expansion of m_κ^2 in ϵ_κ^2 as mentioned above and make a series expansion for small corrections δm_κ^2, where we only keep the leading term of $O\left((\delta m_\kappa^2)^0\right)$:

$$\partial_\kappa g_\kappa^2 = \frac{24g_\kappa^4 \kappa^4 T}{\left(\kappa^2 + m^2\right)^3 \pi^2} . \tag{116}$$

Separation of the variables and integration of both sides from κ' to ξ yields:

$$-\frac{1}{g_\kappa^2}\bigg|_{\kappa'}^{\xi} = \frac{3T}{\pi^2}\left(\frac{3}{m}\arctan\left(\frac{\kappa}{m}\right) - \frac{5\kappa^3 + 3\kappa m^2}{\left(\kappa^2 + m^2\right)^2}\right)\Bigg|_{\kappa'}^{\xi} + O\left(\xi/(gT)\right) . \tag{117}$$

To obtain the general solution for g_κ^2 we have to insert the limits of the integration in the latter equation where we only mention the asymptotic behavior of $\xi \to \infty$. Of course g_ξ^2 is simply g^2 and the solution is expanded in g:

$$g_\kappa^2 = g^2 + \frac{9g^3}{\pi^2}\left(\arctan\frac{\kappa}{gT} - \frac{\pi}{2} - \frac{g\kappa T}{3}\frac{5\kappa^2 + 3g^2T^2}{\left(\kappa^2 + g^2T^2\right)^2} + \frac{8gT}{3\xi} + O\left(\left(gT/\xi^2\right)^2\right)\right) . \tag{118}$$

In this formula, we insert the result for m in first order, $m = gT$, which we already know from perturbation theory. For $\kappa = 0$ we obtain the simple result

$$g_\kappa^2 = g^2 - \frac{9}{2\pi} g^3 + O\left(g^3 \xi / T\right) . \tag{119}$$

It is interesting to mention that the result is independent of the temperature up to order g^3. The temperature can merely change the renormalization scale, which only happens at order g^4 or higher. This is why the results from perturbation theory and the NPRG-formalism are the same up to g^3 and start to differ from g^4.

6.5 Thermal Mass at Finite Temperature

In order to compute the thermal mass at finite temperature we have to split it into two pieces, the vacuum piece $m^2_{vac,\kappa} = m^2_\kappa(T=0)$, and the thermal one, $m^2_{th,\kappa} = m^2_\kappa(T) - m^2_{vac,\kappa}$. We also split the coupling into g^2_κ and $g^2_{vac,\kappa}$. At every order, we have to subtract the vacuum contribution from the full thermal mass. If we omit this, the results will always diverge because of the infinite vacuum contribution. This means that we subtract the corresponding flow equation of the zero-temperature flow from the flow equation of the thermal mass. The thermal mass can be calculated by integrating the new differential equation from the cut-off Λ to κ':

$$m^2_{th,\kappa} = -\frac{1}{\pi^2} \int_\Lambda^\kappa d\kappa\, \kappa'^4 \left\{ g^2_{\kappa'} \frac{1 + 2n(\epsilon_{\kappa'}) - 2\epsilon_{\kappa'} n'(\epsilon_{\kappa'})}{\epsilon^3_{\kappa'}} - g^2_{vac,\kappa'} \frac{1}{\left(\kappa'^2 + m^2_{vac,\kappa'}\right)^{3/2}} \right\} . \tag{120}$$

6.5.1 Leading Order

The thermal mass at leading order is, as it can be seen by numerical calculations, dominated by hard momenta. Thus we are working in the $\kappa \gtrsim \xi$ regime. Since we can use $\epsilon^2_\kappa = \kappa^2$ and $g^2_\kappa = g^2_{vac,\kappa} = g^2$, the vacuum part cancels with the factor $\frac{g^2}{\kappa^3}$ of the first part of Eq. (120):

$$m^2_{th,\kappa} = \frac{2g^2}{\pi^2} \int_\kappa^\Lambda d\kappa'\, \kappa' \cdot \{n(\kappa') - \kappa' n'(\kappa')\} + O\left(g^4\right) . \tag{121}$$

Although we have calculated this expression for $\kappa \gtrsim \xi$, we are allowed to set $\kappa = 0$ and $\Lambda \to \infty$ since the error for κ is of order $O\left(g^2 \xi T\right)$ and the one for Λ is infinitely small as well for $n(\kappa)$ as for $n'(\kappa)$. Afterward, we can use partial integration since the surface term vanishes in the limits $\kappa \to 0$ and $\Lambda \to \infty$. This trick will be used more often, therefore we want to show it once explicitly:

$$\begin{aligned} m^2_{th,0} &= \frac{2g^2}{\pi^2} \int_0^\infty d\kappa\, \kappa \cdot \{n(\kappa) - \kappa n'(\kappa)\} \\ &= \frac{2g^2}{\pi^2} \left[\int_0^\infty d\kappa\, \kappa n(\kappa) - \overbrace{\kappa^2 n(\kappa)\big|_0^\infty}^{=0} + 2 \int_0^\infty d\kappa\, \kappa n(\kappa) \right] \\ &= \frac{6g^2}{\pi^2} \int_0^\infty d\kappa\, \kappa n(\kappa) . \end{aligned} \tag{122}$$

The resulting integral can be solved and yields the expected result for the screening mass at $O\left(g^2\right)$:

$$m^2_{th,0}\big|_{g^2} = m^2 = g^2T^2\,. \tag{123}$$

In fact, it is also possible to compute the behavior of the thermal mass for arbitrary, but small, κ. We start with the following equation, which is pretty the same as above just with the old limits of integration:

$$m^2_{th,\kappa} \;=\; \frac{2g^2}{\pi^2}\left[3\int_\kappa^\Lambda d\kappa'\;\left\{\kappa' n(\kappa') - \kappa'^2 n(\kappa')\right\}\right]. \tag{124}$$

Now we perform a Taylor series expansion for $\kappa n(\kappa)$ and also for $\kappa^2 n(k)$ which is the same expression multiplied once again with κ :

$$\kappa n(\kappa) \simeq T - \frac{\kappa}{2} + \frac{\kappa^2}{12T} + ... \tag{125}$$

Now we are able to integrate each term separately. Since this expansion is only valid for small κ, we split our integration into two ranges and take the limit $\Lambda \to \infty$:

$$\begin{aligned}
m^2_{th,\kappa} &= \frac{2g^2}{\pi^2}\lim_{\epsilon\to 0}\left[3\int_\kappa^\epsilon d\kappa'\left\{\kappa' n(\kappa') - \kappa'^2 n(\kappa')\right\} + 3\int_\epsilon^\infty d\kappa'\left\{\kappa' n(\kappa') - \kappa'^2 n(\kappa')\right\}\right].\\
&= g^2T^2 + \frac{2g^2}{\pi^2}\lim_{\epsilon\to 0}\left[3\int_\kappa^\epsilon d\kappa'\left\{T - \frac{\kappa'}{2}\right\} - \left\{\kappa' T - \frac{\kappa'^2}{2}\right\}\Bigg|_\kappa^\epsilon\right] \qquad (126)\\
&= g^2T^2 + \frac{2g^2}{\pi^2}\lim_{\epsilon\to 0}\left[3\epsilon T - 3\kappa T - \frac{3\epsilon^2}{4} + \frac{3\kappa^2}{4} - \epsilon T + \frac{\epsilon^2}{2} + \kappa T - \frac{\kappa^2}{2}\right]\\
&= g^2T^2 + \frac{2g^2}{\pi^2}\left[-2\kappa T + \frac{\kappa^2}{4}\right]\\
&= g^2\left(T^2 - \frac{4T\kappa}{\pi^2} + \frac{\kappa^2}{2\pi^2} + O\left(\kappa^5\right)\right). \qquad (127)
\end{aligned}$$

In the first line we take the limit $\epsilon \to 0$ for the second integral and recover the result for $\kappa = 0$, $m^2 = g^2T^2$. After performing the integration in the second line we take the same limit for the second integral as before.

6.5.2 Next-to-Leading Order

If we want to calculate the clean $O\left(g^3\right)$ contribution of the screening mass, we start once again with the unexpanded integral, Eq. (120), and subtract the $O\left(g^2\right)$ contribution.

$$\begin{aligned}
m^2_{th,\kappa'} = \frac{1}{\pi^2}\int_{\kappa'}^\Lambda d\kappa\,\kappa^4 \Bigg\{ & g_\kappa^2\frac{1 + 2n(\epsilon_\kappa) - 2\epsilon_\kappa n'(\epsilon_\kappa)}{\epsilon_\kappa^3} - g^2_{vac,\kappa}\frac{1}{\left(\kappa^2 + m^2_{vac,\kappa}\right)^{3/2}}\\
& -2g^2\frac{n(\kappa) - \kappa n'(\kappa)}{\kappa^3}\Bigg\} + O\left(g^4\right).
\end{aligned} \tag{128}$$

For hard momenta, $\kappa \gtrsim \xi$, we can use our expansion of $\epsilon_\kappa \approx \epsilon_{vac,\kappa} \approx \kappa$ in the denominators and in the numerators because of the similar behavior of the thermal and the vacuum flow of the mass. Therefore, all the expressions cancel and the integral vanishes up to order g^4.

For the soft momenta, in the regime $\kappa \lesssim \xi$, we are allowed to expand the thermal distribution function.

$$n(\epsilon_\kappa) - \epsilon_\kappa n'(\epsilon_\kappa) \approx \frac{2T}{\epsilon_\kappa} \,. \tag{129}$$

For the expansion of the distribution functions of the g^2 contribution, we just have to replace ϵ_κ by κ. In order to obtain the g^3 contribution, it is enough to take the quadratic term of the coupling, $g_\kappa^2 = g^2 = g_{vac,\kappa}^2$, and the first order of the mass, $m_\kappa^2 = m^2 = g^2T^2$. With these approximations, the integral becomes very simple. The vacuum contribution cancels the $g_\kappa^2 \frac{1}{\epsilon_\kappa^3}$-term from the first part and we work again in the limit $\kappa \to 0$ and $\xi \to \infty$:

$$m_{th,\kappa}^2 = \frac{4g^2T}{\pi^2} \int_0^\infty d\kappa \left\{ \frac{\kappa^4}{(\kappa^2+m^2)^2} - 1 \right\} = -\frac{3Tm}{\pi} g^2 = -\frac{3T^2}{\pi} g^3 \,. \tag{130}$$

To sum up, we can write the thermal mass as

$$m_{th,0}^2 = g^2T^2 - \frac{3T^2}{\pi} g^3 + O\left(g^4 \ln g\right) \,, \tag{131}$$

which is exactly the same result obtained earlier in perturbation theory. The latter integral can be solved analytically and expanded around $\xi \to \infty$, this yields the general expression for arbitrary κ:

$$m_{th,\kappa}^2|_{g^3} = \frac{6g^3T^2}{\pi^2} \left(\arctan \frac{\kappa}{gT} - \frac{\pi}{2} - \frac{1}{3} \frac{g\kappa T}{\kappa^2 + g^2T^2} + \frac{4gT}{3\xi} + O\left((gT/\xi)^2 \right) \right) \,. \tag{132}$$

7 Thermal Mass at Higher Orders

The calculation of the thermal mass at order g^4 and $g^4 \ln g$ turns out to be difficult. We start, as usual, with the two energy scales introduced earlier. After computing the mass for $\kappa \lesssim \xi$, we try to obtain a result for the other regime, which matches well with the first one.

7.1 Soft Momenta

In the derivation of the screening mass, we completely neglected the flow of g_κ^2 and m_κ^2 in the first two terms of the unexpanded flow equation, Eq. (128). The third term is the $O\left(g^2\right)$ contribution that is subtracted in order to obtain the clean $O\left(g^3\right)$ contribution, so the flow of g_κ^2 has no influence on this term. This allows us to write the flow of the coupling as a main contribution g^2 and a correction δg_κ^2, which is simply given by the $O\left(g^3\right)$ contribution of g_κ^2:

$$\delta g_\kappa^2 = \frac{9g^3}{\pi^2}\left(\arctan\frac{\kappa}{gT} - \frac{\pi}{2} - \frac{g\kappa T}{3}\frac{5\kappa^2+3g^2T^2}{\left(\kappa^2+g^2T^2\right)^2} + \frac{8gT}{3\xi}\right) . \tag{133}$$

As described in the latter chapter, we use $\epsilon_\kappa^2 = \epsilon_{vac,\kappa}^2$. This means that we neglect higher order terms of m_κ^2 and $m_{vac,\kappa}^2$. Now we want to compute the correction δm_κ^2 to the screening mass, so we simply insert $g_\kappa^2 = g^2 + \delta g_\kappa^2$ in the flow equation, which yields:

$$\delta m_{th,0}^2 = \frac{4T}{\pi^2}\int_0^\xi d\kappa\, \delta g_\kappa^2 \cdot \frac{\kappa^4}{\left(\kappa^2+m^2\right)^2} . \tag{134}$$

Since δg_κ^2 contains different kinds of terms, we have to integrate every part separately. The solution would contain terms proportional to $\ln\left(m^2+\xi^2\right)$. Since we want to take the limit $\xi \to \infty$ later on, the mass is neglectable in these terms. Additionally, terms of higher order in g will appear that can be neglected too. In the end, we obtain the correction:

$$\delta m_{th,0}^2 = \frac{96g^4T^2}{\pi^4}\left[\ln\frac{gT}{\xi} + \frac{25}{24} + \frac{9\pi^2}{128} - \frac{3\pi gT}{4\xi} + O\left(\left(gT/\xi\right)^2\right)\right] . \tag{135}$$

The problem is hidden in the logarithmic term, $\ln\frac{gT}{\xi}$. In the limit $\xi \to \infty$ this term would diverge, therefore we would expect to find a corresponding term proportional to $\ln\frac{\xi}{T}$ in the UV region, in order to cancel the divergence.

7.2 Hard Momenta

The calculation of the screening mass at order $O(g^4)$ for the $\kappa \gtrsim \xi$ regime is more complicated than one would expect after the calculation in the other regime. In a first attempt, we expand $\epsilon_\kappa^2 = \kappa^2\left(1+O\left(g^2T^2/\kappa^2\right)\right)$ for the thermal part as well as for the vacuum part and insert the flow of the coupling in this regime, which is equal to the flow of the vacuum coupling, into the unexpanded flow equation for the mass:

$$g_\kappa^2 = g^2 + \frac{9g^4}{2\pi^2}\ln\frac{\kappa}{2\pi T} . \tag{136}$$

Since there is no g^3 contribution, it is still enough to subtract the clean g^2 contribution.

$$\begin{aligned}
m_{th,0}^2\Big|_{g^4} &= \frac{1}{\pi^2}\int_0^\xi d\kappa\,\kappa^4\Big\{g_\kappa^2\frac{1+2n(\epsilon_\kappa)-2\epsilon_\kappa n'(\epsilon_\kappa)}{\epsilon_\kappa^3} - g_{vac,\kappa}^2\frac{1}{\left(\kappa^2+m_{vac,\kappa}^2\right)^{3/2}} \\
&\quad -2g^2\frac{n(\kappa)-\kappa n'(\kappa)}{\kappa^3}\Big\} \\
&= \frac{1}{\pi^2}\int_0^\xi d\kappa\,\kappa^4\Big\{\left(g^2+\frac{9g^4}{2\pi^2}\ln\frac{\kappa}{2\pi T}\right)\frac{1+2n(\kappa)-2\kappa n'(\kappa)}{\kappa^3} \\
&\quad -\left(g^2+\frac{9g^4}{2\pi^2}\ln\frac{\kappa}{2\pi T}\right)\frac{1}{\kappa^3} - 2g^2\frac{n(\kappa)-\kappa n'(\kappa)}{\kappa^3}\Big\} \\
&= \frac{9g^4}{\pi^4}\int_0^\xi d\kappa\,\ln\frac{\kappa}{2\pi T}\cdot\left(\kappa n(\kappa)-\kappa^2 n'(\kappa)\right)\,. \qquad (137)
\end{aligned}$$

Due to the appearance of the distribution function, the latter integral has no analytical expression. One possibility is to extend the integration interval to infinity. Since the logarithmic function and the expressions κ and κ^2 increase slower than the exponential function in the denominator goes to zero, the error still vanishes. Then, the integral is solvable numerically. The problem is, that we will not be able to find a term proportional to $\ln\xi$ in this way.

$$m_{th,0}^2\Big|_{g^4} = -0.301692\,g^4T^2 - 0.451402\,g^4T^2 = -0.753094\,g^4T^2\,. \qquad (138)$$

The second possibility is to proceed as in the chapter before: Even in the regime of hard momenta, we consider κ to be small enough to expand the part including the distribution function and its derivative. Since

$$\kappa n(\kappa) - \kappa^2 n'(\kappa) \approx 2T - \frac{\kappa}{2} + \frac{\kappa^4}{360T^3} + \dots \qquad (139)$$

we have to deal with the following integral:

$$m_{th,0}^2\Big|_{g^4} = \tfrac{9g^4}{\pi^4}\int_\xi^\Lambda d\kappa\,\left\{\ln\tfrac{\kappa}{2\pi T}\cdot\left(2T-\tfrac{\kappa}{2}+\tfrac{\kappa^4}{360T^3}\right)\right\}\,. \qquad (140)$$

Unfortunately, the algorithm breaks down here: At first, we can proceed as before and split the integral into two integration ranges. In the remaining result, every term is proportional to ξ, which should not be the case. To remember: We are looking for a term $\propto g^4T^2\ln\xi$.

In order to improve our calculations, we check if the approximation $\epsilon_\kappa^2 = \kappa^2$ is too strict. As seen in Fig. 2, m_κ and $m_{\kappa,vac}$ show the same behavior in this regime. Therefore, we try to expand ϵ_κ^2 further than we have done it so far ($\epsilon_\kappa^2 \approx \kappa^2$):

$$\epsilon_\kappa^2 \approx \kappa^2\left(1+\frac{g^2}{2\pi^2}\right)\,. \qquad (141)$$

In this equation we insert the first order of the vacuum flow of the mass: $m_{\kappa,vac}^2 = \frac{\kappa^2 g^2}{2\pi^2}$. ϵ_κ can be expanded for small g:

$$\epsilon_\kappa = \kappa\sqrt{1+\frac{g^2}{2\pi^2}} \approx \kappa\left(1+\frac{g^2}{4\pi^2}\right) . \tag{142}$$

Also the distribution function depends on ϵ_κ that is now proportional to the coupling, so we perform once again an expansion for small g:

$$n(\epsilon_\kappa) = n(\kappa) + n'(\kappa)\frac{g^2}{2\kappa} + ... \tag{143}$$

For a better overview, we also calculate the expansion of ϵ_κ^{-3} here:

$$\frac{1}{\epsilon_\kappa^3} \approx \frac{1}{\kappa^3}\left(1-\frac{3g^2}{4\pi^2}\right) . \tag{144}$$

Inserting all these expressions into the unexpanded equation yields the following integral:

$$\begin{aligned} m_{th}^2\big|_{g^4} \quad = \quad & \frac{1}{\pi^2}\int_\xi^\Lambda d\kappa\,\kappa^4\Bigg\{\left(g^2+\frac{9g^4}{2\pi^2}\ln\frac{\kappa}{2\pi T}\right)\frac{1}{\kappa^3}\left(1-\frac{3}{4}\frac{g^2}{\pi^2}\right) \\ & \cdot\left[1+2n(\kappa)+\frac{g^2}{\kappa}n'(\kappa)-\kappa\left(1+\frac{g^2}{4\pi^2}\right)\left(n'(\kappa)-n'(\kappa)\frac{\left(1+e^{\kappa/T}\right)\kappa g^2}{4\pi^2 T}\right)\right] \\ & -\left(g^2+\frac{9g^4}{2\pi^2}\ln\frac{\kappa}{2\pi T}\right)\frac{1}{\kappa^3}\left(1-\frac{3}{4}\frac{g^2}{\pi^2}\right)-2g^2\frac{n(\kappa)-\kappa n'(\kappa)}{\kappa^3}\Bigg\} . \end{aligned} \tag{145}$$

It can be simplified by multiplying all the braces out and ignoring all terms proportional to higher orders of g. Additionally, the g^2 contribution and the vacuum divergences cancel out.

$$\begin{aligned} m_{th}^2\big|_{g^4} \quad = \quad & \frac{1}{\pi^2}\int_\xi^\Lambda d\kappa\,\kappa\Bigg\{\left(-\frac{3}{2}\frac{g^2}{\pi^2}+\frac{9g^4}{\pi^2}\ln\frac{\kappa}{2\pi T}\right)[n(\kappa)-\kappa n'(\kappa)] \\ & +g^4\left[\frac{n'(\kappa)}{\kappa}-\frac{\kappa}{2\pi^2}n'(\kappa)+2\kappa^2 n'(\kappa)\frac{1+e^{\kappa/T}}{4\pi^2 T}\right]\Bigg\} \\ = \quad & \frac{1}{\pi^2}\int_\xi^\Lambda d\kappa\,\frac{9g^4}{\pi^2}\kappa\ln\frac{\kappa}{2\pi T}[n(\kappa)-\kappa n'(\kappa)] \\ & +\frac{1}{\pi^4}g^4\int_\xi^\Lambda d\kappa\left[-\frac{3\kappa}{2}n(\kappa)+\kappa^2 n'(\kappa)+\kappa^3 n'(\kappa)\frac{1+e^{\kappa/T}}{2T}\right] . \end{aligned} \tag{146}$$

In the third line we were able to reproduce the expression of Eq. (137), which should be included in this calculation as first order contribution. All these expressions include the distribution function. Therefore, we have to expand the limits of integration once again to zero and infinity. If we are able to calculate these integrals in these limits, we show that these expressions are not proportional to $\ln\xi$ either. Only if the integrals diverge, they can lead to such a term.

For the first term, we insert the result from the beginning of this chapter. The first two terms of the last line can be computed using partial integration, the last term, which includes

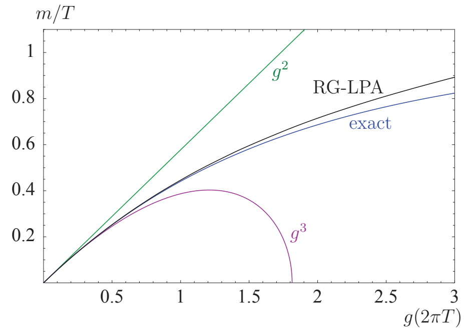


Figure 3: Comparison of the mass obtained in the NPRG-formalism with the results of the large N limit. The coupling g denotes the effective coupling $g_{eff} = g\sqrt{N}$ in the limit $N \to \infty$ [1].

the exponential factor, can also be calculated in these limits.

$$
\begin{aligned}
m_{th}^2\big|_{g^4} &= -0.753094\, g^4T^2 - \frac{5g^4}{2\pi^2}\frac{T^2}{6} - \frac{6g^4T^2}{\pi^4}\zeta(3) + \frac{g^4T^2}{30}\,, \\
m_{th}^2\big|_{g^4} &= -0.836021\, g^4T^2\,.
\end{aligned} \tag{147}
$$

$\zeta(3)$ is the Riemann-Zeta function evaluated at 3. This result shows that the more accurate expansion of $\epsilon_\kappa^2 = \kappa^2\left(1 + \frac{g^2}{2\pi^2}\right)$ really contributes to the thermal mass, but does not yield the desired $\ln\xi$ term. One reason could be that the derivation of the flow equations for the vacuum and the thermal flow ignore the higher order terms of this expansion. It is therefore questionable, whether the unexpanded flow equation, which we use in this calculation, is still valid in this regime at order g^4. A new derivation, which includes these terms, would contain many additional terms that can not be integrated analytically and is therefore out of the scope of this thesis.

7.3 Comparison of the Results with Perturbation Theory

Although we were not able to compute the complete result at order g^4 and $g^4 \ln g$, we can compare these results with perturbation theory. In order to do this, we have to sum up the results as well for $\kappa \lesssim \xi$ as for $\kappa \gtrsim \xi$ at order g^4. At order $g^4 \ln g$, we assume that the soft momenta produce all the contributing terms and the hard momenta only cancel the $\ln\xi$.

	NPRG	PE
g^4T^2	0.8745	-0.405
$g^4 \ln g$	0.9855	-0.608

If we are able to change the renormalization scheme in such a way that the results get closer, we would have to subtract something from the results calculated within the NPRG-formalism. This fits well with the numerical results from [1], plottet in Fig. 3. RG-LPA denotes the numerical results coming from the NPRG within the local potential approximaion. The exact result is obtained in a large N limit calculation. A suitable change of the renormalization scheme, which converges the results of the two methods at order g^4 and $g^4 \ln g$, would result in

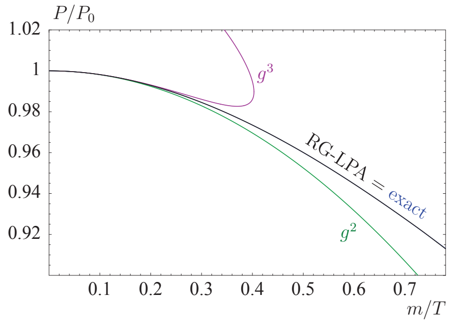


Figure 4: Pressure as a function of the screening mass. In the comparis of physical quantities, the results of NPRG-formalism fit perfectly to the results of the large N limit [1].

an improvement of the whole result of the NPRG-formalism for the mass. This shows, that the latter calculations that we have done are heading into the right direction, even if they do not cover the whole result.

In order to understand how we could maybe change the renormalization scheme, we go back to Eq. (110) and Eq. (111):

$$m_\Lambda^2 - m_\kappa^2 = -\frac{g^2}{2\pi^2}\left(\Lambda^2 - \kappa^2\right) , \tag{148}$$

$$m_\Lambda^2 = -\frac{g_\Lambda^2 \Lambda^2}{2\pi^2} . \tag{149}$$

In principle, it would be possible to change the integration constant. Since the flow of the vacuum mass is considered the first time at order g^4, it would not change the results at lower orders of the coupling, which is very important. The problem is, that the vacuum flow has to vanish for $\kappa = 0$. If not, it would mean that we start with a non vanishing physical mass. That would change all calculations, including the pressure and the coupling as well. Consequently, we are not to allowed to add a term, which is not proportional to κ. Of course, adding a term proportional to κ is not valid either, since it would change the flow equation and is mathematically not correct.

A possibility to cancel out the influence of the renormalization scale is to compare only physical quantities. For example, one could compare the flow of the pressure with the flow of the mass. As it is shown in Fig. 4 , the result fits perfectly with the exact result, obtained again from a large N limit calculation.

8 Conclusion and Outlook

After a recapitulation of the results of perturbation theory, we extracted the thermal mass order by order from the flow equation of the NPRG-formalism, with the aim of comparing them with the perturbative results. Therefore we derived the central equation of the NPRG, the flow equation, and used it within the local potential approximation. Using this approximation, we computed the vacuum flow and the pertrubative results for the coupling, the pressure and the screening mass and the thermal flow for the coupling and the mass. The flow of the coupling shows now dependency of the temperature up to order g^3. This is why the results for the mass fit exactly up to order g^3 and start to differ at g^4 due to the different renormalization schemes, which are influenced by the temperature. Therefore, we computed different contributions of various energy scales to the flow of the thermal mass at order g^4 and $g^4 \ln g$ in the non-pertrubative renormalization group formalism. We explained that the simple splitting of the flow equation into two energy regimes is not enough and used a third scale with a different expansion of ϵ_κ^2. This additional scale contributes to the mass at order g^4. Afterwards we have shown that a simple change of the integration constant would change the result for the mass starting from order g^4, but it would not be a correct way to adapt the renormalization scheme. Moreover, we discussed that the obtained results for the the mass are not so bad at all, since they match qualitatively to the numerical results obtained in literature from the large N limit. [1]

At the regime where the change of the thermal mass is very strong, but not yet similar to the vacuum flow, it could be necessary to introduce a further energy scale, where the flow of the mass at lower orders of g for arbitrary κ has to be taken into account. Maybe the missing $\ln \xi$ term can be found there. In the regime of hard momenta, where we have taken the first order contribution of the flow of the mass to ϵ_κ^2 into account, it would be reasonable to derive the flow equations from the very beginning, including the additional term.

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